CHAPTER THREE

INNER PRODUCT SPACES AND HILBERT SPACES

DEFN. An inner product space X is a space with an inner product $X \times X \to \mathbb{R}$ (or $\mathbb{C}$), satisfying

(i) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle \quad \forall x, y, z \in X$

(ii) $\langle \lambda x, y\rangle = \lambda \langle x, y\rangle \quad \forall x, y \in X$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$)

(iii) $\langle x, y\rangle = \overline{\langle y, x\rangle} \quad \forall x, y \in X$

(iv) $\langle x, x\rangle \geq 0 \quad \forall x \in X$

and $\langle x, x\rangle = 0$ if and only if $x = 0$.

REMARK: (ii) implies that $\langle x, \lambda y\rangle = \bar{\lambda}\langle x, y\rangle \quad \forall x, y \in X$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$).

REMARK: On an inner product space X, define

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

It can be checked that $(X, \|\cdot\|)$ is a normed space.

DEFN. We say that X is a Hilbert space, if X is an inner product space and $(X, \|\cdot\|)$ is complete, where $\|\cdot\|$ is given by $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in X$.

Lemma (Schwarz Inequality, triangle inequality).

An inner product and the corresponding norm satisfy the Schwartz inequality and the triangle inequality as follows:

(a)
(1) $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$. [Schwarz Inequality]

where the equality sign holds if and only if x and y are linear dependent (i.e. $\exists\ \lambda_1, \lambda_2$ s.t. $\lambda_1 x + \lambda_2 y = 0$ and at least one of λ_1 and λ_2 is not zero).

(b)
(2) $\|x + y\| \leq \|x\| + \|y\|$. [Triangle Inequality].

Proof:

(a) If $y = 0$, then (1) holds.

Assume $y \neq 0$, for every $\lambda \in \mathbb{C}$, we have

$$0 \leq \|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y\rangle$$

$$= \langle x, x\rangle - \bar{\lambda}\langle x, y\rangle - \lambda[\langle y, x\rangle - \bar{\lambda}\langle y, y\rangle]$$

Let $\bar{\lambda} = \dfrac{\langle y, x\rangle}{\langle y, y\rangle}$, and it yields

$$0 \leq \langle x, x \rangle - \frac{\langle y, x \rangle}{\langle y, y \rangle} \langle x, y \rangle$$

$$= \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2},$$

which finishes the proof of (1).

(2). It follows from the Schwartz Inequality.

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$= \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2$$

By the above proved Schwartz inequality, we have

$$|\langle x, y \rangle| = |\langle y, x \rangle| \leq \|x\| \cdot \|y\|.$$

Thus

$$\|x+y\|^2 \leq \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2,$$

which finishes the proof. $\square$

From this, we can show that $\langle x, \cdot \rangle$ is continuous for every $x \in M$, where M is an inner product space.

<u>Lemma</u>: In a pre-Hilbert space M, $\forall x \in M$.

$$\langle x, \cdot \rangle : M \longrightarrow \mathbb{C}$$

$$y \mapsto \langle x, y \rangle$$

is continuous.

<u>Proof</u>:

It follows directly from the fact that

$$\| \langle x, y \rangle \| \leq \|x\| \cdot \|y\|. \qquad \square$$

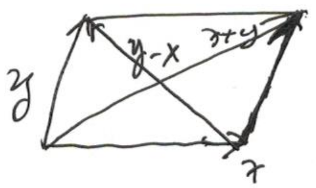
Prop. (~~Parallel~~ Parallelogram Equality)

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

where x, y are arbitrary elements in inner product space M, and the norm $\|x\|$ is defined to be $\sqrt{\langle x, x\rangle}$, $\forall x \in M$.

Proof: Exercise.

Remark:



From the prop. above, it is NOT hard to check/reach the following fact:

FACT: Not all normed spaces are inner product spaces.

E.g. l^1 is NOT an inner product space.

Inner product spaces are also called pre-Hilbert spaces b/c. of the following defn.

DEFN. If an inner product space M is complete (under the norm induced by the inner product), then we say M is a Hilbert space.

DEFN (Orthogonality).

An element x of an inner product space X is said to be orthogonal to an element $y \in X$ if

$$\langle x, y \rangle = 0 .$$

This is also denoted as $x \perp y$.

Similarly, for two subsets A and B of X, we say $A \perp B$ if $x \perp y$ $\forall x \in A$ and $y \in B$.

Exercise: The space ℓ^p (with $p \neq 2$) is NOT an inner product space, hence not a Hilbert Space.

Exercise: The Banach space $(C[0,1], \|\cdot\|)$ is NOT an inner product space, hence not a Hilbert space. Here $\|\cdot\|$ is defined as

$$\|f\| = \sup_{x \in [0,1]} f(x) \overset{\text{as } [0,1] \text{ is compact}}{=\!=} \max_{x \in [0,1]} f(x) .$$

On an inner product space M, from inner product structure $\langle \, , \, \rangle$, we can induce the norm structure. What about the inverse? In other words, if we know that a norm is induced from an inner product $\langle \, , \rangle$. via $\|x\| = \sqrt{\langle x, x\rangle}$.

Can we trace back to the original inner product? Is it possible that two inner products might yield the same norm?

Exercise: For a real ($\mathbb{R}$) inner product space, we have

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right).$$

This is the link between inner products and norms derived from it.

Exercise: If $x \perp y$ in an inner product space, show that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

(Pythagorean Identity).

Exercise: In an inner product space, if $\langle x, u\rangle = \langle x, v\rangle \ \forall x$, show that $u = v$.